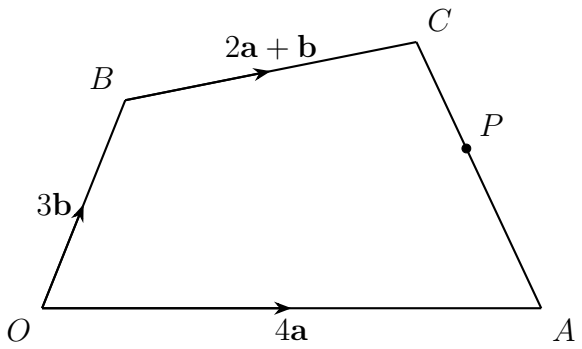
Vectors: Geometric Proofs & Line Intersections

Practice Worksheet (Jun 2024 Paper 2H Style)

I Do



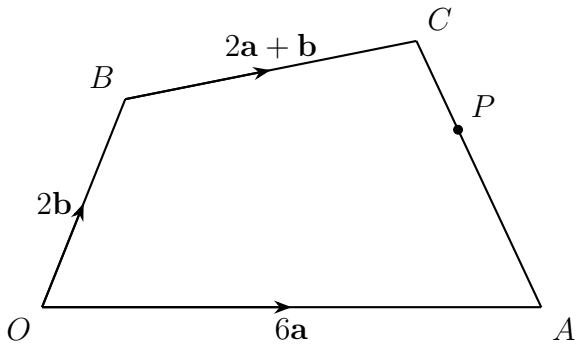
The diagram shows a quadrilateral $OACB$ in which $\overrightarrow{OA} = 4\mathbf{a}$, $\overrightarrow{OB} = 3\mathbf{b}$, and $\overrightarrow{BC} = 2\mathbf{a}+\mathbf{b}$.

(a) Find $\overrightarrow{AC}$ in terms of $\mathbf{a}$ and $\mathbf{b}$. Give your answer in its simplest form.

The point P lies on AC such that $AP : PC = 3 : 2$.
The point Q is such that OPQ and BCQ are straight lines.

(b) Using a vector method, find $\overrightarrow{OQ}$ in terms of $\mathbf{a}$ and $\mathbf{b}$. Give your answer in its simplest form. Show your working clearly.

We Do



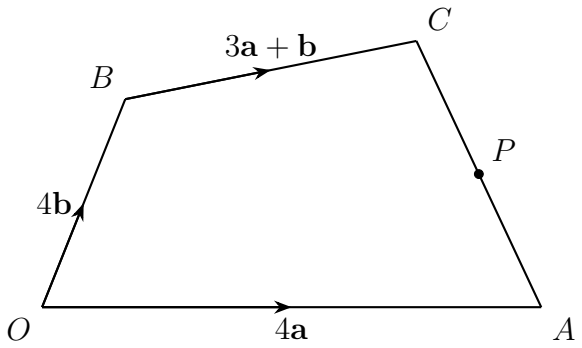
The diagram shows a quadrilateral $OACB$ in which $\overrightarrow{OA} = 6\mathbf{a}$, $\overrightarrow{OB} = 2\mathbf{b}$, and $\overrightarrow{BC} = 2\mathbf{a}+\mathbf{b}$.

(a) Find $\overrightarrow{AC}$ in terms of $\mathbf{a}$ and $\mathbf{b}$. Give your answer in its simplest form.

The point P lies on AC such that $AP : PC = 2 : 1$.
The point Q is such that OPQ and BCQ are straight lines.

(b) Using a vector method, find $\overrightarrow{OQ}$ in terms of $\mathbf{a}$ and $\mathbf{b}$. Give your answer in its simplest form. Show your working clearly.

You Do



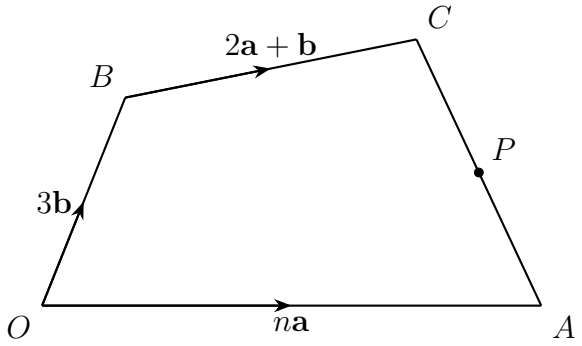
The diagram shows a quadrilateral $OACB$ in which $\overrightarrow{OA} = 4\mathbf{a}$, $\overrightarrow{OB} = 4\mathbf{b}$, and $\overrightarrow{BC} = 3\mathbf{a} + \mathbf{b}$.

(a) Find $\overrightarrow{AC}$ in terms of $\mathbf{a}$ and $\mathbf{b}$. Give your answer in its simplest form.

The point P is the midpoint of AC (so $AP : PC = 1 : 1$).
The point Q is such that OPQ and BCQ are straight lines.

(b) Using a vector method, find $\overrightarrow{OQ}$ in terms of $\mathbf{a}$ and $\mathbf{b}$. Give your answer in its simplest form. Show your working clearly.

Challenge



In the quadrilateral $OACB$ shown above, $\overrightarrow{OA} = n\mathbf{a}$, $\overrightarrow{OB} = 3\mathbf{b}$, and $\overrightarrow{BC} = 2\mathbf{a} + \mathbf{b}$.
P lies on AC such that $AP : PC = 1 : 1$ (so P is the midpoint of AC).
Q is the intersection of the extended lines OP and BC.

Given that $\overrightarrow{OQ} = 6\mathbf{a} + 6\mathbf{b}$, find the value of the constant n.

Answers

- **I Do:**

$$\text{(a) } \overrightarrow{AC} = \overrightarrow{AO} + \overrightarrow{OB} + \overrightarrow{BC} = -4\mathbf{a} + 3\mathbf{b} + (2\mathbf{a} + \mathbf{b}) = \mathbf{-2a + 4b}$$

$$\text{(b) } \overrightarrow{OP} = \overrightarrow{OA} + \frac{3}{5}\overrightarrow{AC} = 4\mathbf{a} + \frac{3}{5}(-2\mathbf{a} + 4\mathbf{b}) = \frac{14}{5}\mathbf{a} + \frac{12}{5}\mathbf{b}$$

$$\text{Since } O, P, Q \text{ are collinear, } \overrightarrow{OQ} = m\overrightarrow{OP} = \frac{14m}{5}\mathbf{a} + \frac{12m}{5}\mathbf{b}$$

$$\text{Since } B, C, Q \text{ are collinear, } \overrightarrow{OQ} = \overrightarrow{OB} + k\overrightarrow{BC} = 3\mathbf{b} + k(2\mathbf{a} + \mathbf{b}) = 2k\mathbf{a} + (3 + k)\mathbf{b}$$

$$\text{Equating coeffs: } 2k = \frac{14m}{5} \implies k = \frac{7m}{5}$$

$$3 + \frac{7m}{5} = \frac{12m}{5} \implies 3 = m \implies k = \frac{21}{5}$$

$$\overrightarrow{OQ} = 3\left(\frac{14}{5}\mathbf{a} + \frac{12}{5}\mathbf{b}\right) = \mathbf{\frac{42}{5}a} + \mathbf{\frac{36}{5}b} \quad (\text{or } 8.4\mathbf{a} + 7.2\mathbf{b})$$

- **We Do:**

$$\text{(a) } \overrightarrow{AC} = -6\mathbf{a} + 2\mathbf{b} + (2\mathbf{a} + \mathbf{b}) = \mathbf{-4a + 3b}$$

$$\text{(b) } \overrightarrow{OP} = 6\mathbf{a} + \frac{2}{3}(-4\mathbf{a} + 3\mathbf{b}) = \frac{10}{3}\mathbf{a} + 2\mathbf{b}$$

$$\overrightarrow{OQ} = m\left(\frac{10}{3}\mathbf{a} + 2\mathbf{b}\right) = 2k\mathbf{a} + (2 + k)\mathbf{b}$$

$$2k = \frac{10m}{3} \implies k = \frac{5m}{3}$$

$$2 + \frac{5m}{3} = 2m \implies 2 = \frac{m}{3} \implies m = 6 \implies k = 10$$

$$\overrightarrow{OQ} = 6\left(\frac{10}{3}\mathbf{a} + 2\mathbf{b}\right) = \mathbf{20a + 12b}$$

- **You Do:**

$$\text{(a) } \overrightarrow{AC} = -4\mathbf{a} + 4\mathbf{b} + (3\mathbf{a} + \mathbf{b}) = \mathbf{-a + 5b}$$

$$\text{(b) } \overrightarrow{OP} = 4\mathbf{a} + \frac{1}{2}(-\mathbf{a} + 5\mathbf{b}) = \frac{7}{2}\mathbf{a} + \frac{5}{2}\mathbf{b}$$

$$\overrightarrow{OQ} = m\left(\frac{7}{2}\mathbf{a} + \frac{5}{2}\mathbf{b}\right) = 3k\mathbf{a} + (4 + k)\mathbf{b}$$

$$3k = \frac{7m}{2} \implies k = \frac{7m}{6}$$

$$4 + \frac{7m}{6} = \frac{5m}{2} = \frac{15m}{6} \implies 4 = \frac{8m}{6} \implies m = 3 \implies k = \frac{7}{2}$$

$$\overrightarrow{OQ} = 3\left(\frac{7}{2}\mathbf{a} + \frac{5}{2}\mathbf{b}\right) = \mathbf{\frac{21}{2}a} + \mathbf{\frac{15}{2}b} \quad (\text{or } 10.5\mathbf{a} + 7.5\mathbf{b})$$

- **Challenge:**

$$\overrightarrow{AC} = -n\mathbf{a} + 3\mathbf{b} + (2\mathbf{a} + \mathbf{b}) = (2-n)\mathbf{a} + 4\mathbf{b}$$

$$\overrightarrow{OP} = n\mathbf{a} + \frac{1}{2}\left((2-n)\mathbf{a} + 4\mathbf{b}\right) = \left(\frac{n+2}{2}\right)\mathbf{a} + 2\mathbf{b}$$

Since $\overrightarrow{OQ} = 6\mathbf{a} + 6\mathbf{b}$ and O, P, Q are collinear:

$$\overrightarrow{OQ} = m\overrightarrow{OP} \implies 6\mathbf{b} = m(2\mathbf{b}) \implies m = 3$$

$$6\mathbf{a} = 3\left(\frac{n+2}{2}\right)\mathbf{a} \implies 6 = \frac{3n+6}{2} \implies 12 = 3n + 6 \implies \mathbf{n = 2}$$