Algebraic Proofs

Practice Worksheet (Edexcel 4MA1 Style)

I Do

Catherine is working on her algebra homework.

Prove algebraically that the difference between the squares of any two consecutive odd numbers is always a multiple of 8.

(Total for Question is 3 marks)

We Do

Catherine claims that the sum of the squares of any two consecutive even numbers is always 4 more than a multiple of 8.

Prove algebraically that **Catherine** is correct.

(Total for Question is 3 marks)

You Do 1

John needs to prove a mathematical statement about odd numbers.

Prove algebraically that the square of any odd number is always 1 more than a multiple of 8.

(Total for Question is 4 marks)

You Do 2

Ryan states that the difference between the squares of any two consecutive integers is always equal to the sum of those two integers.

Prove algebraically that **Ryan's** statement is true.

(Total for Question is 3 marks)

You Do 3

Paul says that if you add any three consecutive even numbers together, the result is always a multiple of 6.

Prove algebraically that **Paul** is correct.

(Total for Question is 3 marks)

Challenge 1

Sophie notices a pattern with odd numbers.

Prove algebraically that the difference between the squares of **any** two odd numbers (not necessarily consecutive) is always a multiple of 8.

(Total for Question is 4 marks)

Challenge 2

Paige is exploring products of numbers.

Prove algebraically that the product of two consecutive even integers is always a multiple of 8.

(Total for Question is 3 marks)

Extreme Challenge

Katie is investigating a complex integer sequence.

Prove algebraically that the product of four consecutive positive integers is always exactly one less than a square number.
(Hint: Let the four consecutive integers be $n, (n+1), (n+2)$*, and* $(n+3)$*.)*

You must show all your algebraic working clearly.

(Total for Question is 5 marks)

Worked Answers

I Do

Let the two consecutive odd numbers be $(2n-1)$ and $(2n+1)$.
Square both numbers: $(2n+1)^2$ and $(2n-1)^2$.
Difference between the squares $= (2n+1)^2 - (2n-1)^2$.
Expand both brackets:
$(4n^2+4n+1) - (4n^2-4n+1)$.
Simplify by subtracting:
$4n^2+4n+1-4n^2+4n-1 = 8n$.
Since the result is $8n$, which can be written as $8 \times n$, it is always a multiple of 8.

We Do

Let the two consecutive even numbers be $2n$ and $2n+2$.
Sum of their squares $= (2n)^2 + (2n+2)^2$.
Expand the brackets:
$4n^2 + (4n^2+8n+4)$.
Simplify by collecting like terms:
$8n^2+8n+4$.
Factorise out the 8 from the first two terms:
$8(n^2+n)+4$.
Since $8(n^2+n)$ is a multiple of 8, the expression is exactly 4 more than a multiple of 8. Catherine is correct.

You Do 1

Let an odd number be $2n+1$.
Square the odd number: $(2n+1)^2$.
Expand the bracket: $4n^2+4n+1$.
Factorise out 4 from the first two terms: $4n(n+1)+1$.
Notice the term $n(n+1)$. Because n and $n+1$ are consecutive integers, one of them must be even.
Therefore, their product $n(n+1)$ must be an even number (a multiple of 2).
Let $n(n+1) = 2k$ (where k is an integer).
Substitute this back: $4(2k)+1 = 8k+1$.
Since $8k$ is a multiple of 8, the result is always 1 more than a multiple of 8.

You Do 2

Let the two consecutive integers be n and $n+1$.
Difference of their squares $= (n+1)^2 - n^2$.
Expand the bracket: $(n^2+2n+1) - n^2$.
Simplify: $2n+1$.
Now calculate the sum of the two integers:
$n + (n+1) = 2n+1$.
The difference of the squares $(2n+1)$ is exactly equal to the sum of the integers $(2n+1)$. Ryan is correct.

You Do 3

Let the three consecutive even numbers be $2n$, $2n+2$, and $2n+4$.
Add them together: $2n + (2n+2) + (2n+4)$.
Collect like terms: $6n + 6$.
Factorise out the 6: $6(n+1)$.
Since the result is a multiple of 6 (it has a factor of 6), Paul is correct.

Challenge 1

Let any two odd numbers be $2m+1$ and $2n+1$.
Difference of their squares $= (2m+1)^2 - (2n+1)^2$.
Expand both brackets:
$(4m^2 + 4m + 1) - (4n^2 + 4n + 1)$.
Simplify: $4m^2 + 4m - 4n^2 - 4n$.
Factorise terms involving m and terms involving n:
$4m(m+1) - 4n(n+1)$.
As shown in previous proofs, $m(m+1)$ is the product of two consecutive integers, so it is even.
Therefore, $4m(m+1)$ is $4 \times$ even, which is a multiple of 8.
Similarly, $4n(n+1)$ is a multiple of 8.
The difference between two multiples of 8 is always a multiple of 8.

Challenge 2

Let the two consecutive even integers be $2n$ and $2n+2$.
Product $= 2n(2n+2)$.
Factorise out the 2 from the second bracket:
$= 2n \times 2(n+1)$.
$= 4n(n+1)$.
Because n and $n+1$ are consecutive integers, one of them must be even.
Therefore, their product $n(n+1)$ is a multiple of 2 (an even number).
So we have $4 \times$ (a multiple of 2), which is always a multiple of 8.

Extreme Challenge

Let the four consecutive positive integers be n, $(n+1)$, $(n+2)$, and $(n+3)$.
Their product is $P = n(n+1)(n+2)(n+3)$.
Group the outer pair and the inner pair of terms:
$P = [n(n+3)] \times [(n+1)(n+2)]$.
Expand both square brackets:
$P = (n^2 + 3n)(n^2 + 3n + 2)$.
Let $x = n^2 + 3n$. We can substitute x into the expression:
$P = x(x+2)$.
Expand: $P = x^2 + 2x$.
Add and subtract 1 to complete the square:
$P = (x^2 + 2x + 1) - 1$.
$P = (x+1)^2 - 1$.
Substitute x back in: $P = (n^2 + 3n + 1)^2 - 1$.
Since n is an integer, $(n^2 + 3n + 1)$ is an integer, making $(n^2 + 3n + 1)^2$ a perfect square.
Therefore, the product is exactly one less than a square number.