Quadratic Simultaneous Equations

Practice Worksheet (Problem Solving)

I Do

Solve the simultaneous equations.
Give your solutions in exact form, showing clear algebraic working.

$$2x^2 + y^2 = 30$$
$$y = 2x - 3$$

.....

(Total for Question is 5 marks)

We Do

Solve the simultaneous equations.
Give your solutions in exact form, showing clear algebraic working.

$$x^2 + y^2 = 12$$
$$y = x - 2$$

.....

(Total for Question is 5 marks)

You Do 1

Solve the simultaneous equations.
Show clear algebraic working.

$$x^2 + y^2 = 25$$
$$y - 2x = 5$$

.....

(Total for Question is 5 marks)

You Do 2

Solve the simultaneous equations.
Give your solutions as exact integers or fractions, showing clear algebraic working.

$$x^2 + 2y^2 = 18$$
$$x = 2y - 2$$

.....

(Total for Question is 5 marks)

You Do 3

Solve the simultaneous equations.
Give your solutions in exact surd form, showing clear algebraic working.

$$x^2 + y^2 = 20$$
$$y = 2x - 2$$

.....

(Total for Question is 5 marks)

Challenge 1

Solve the simultaneous equations.
Show all of your algebraic working clearly.

$$x^2 + xy = 12$$
$$2x + y = 7$$

.....

(Total for Question is 5 marks)

Challenge 2

The straight line $y = x + 1$ intersects the circle $x^2 + y^2 = 25$ at two points, A and B.

(a) Find the coordinates of A and B.

$A = (\ldots\ldots, \ldots\ldots)$ $B = (\ldots\ldots, \ldots\ldots)$ *(4 marks)*

(b) Calculate the exact length of the line segment AB.
Give your answer in its simplest surd form.

..... *(2 marks)*

(Total for Question is 6 marks)

Extreme Challenge

The straight line $y = 2x + k$ is a tangent to the circle $x^2 + y^2 = 20$, where $k > 0$.

(a) Show that $5x^2 + 4kx + (k^2 - 20) = 0$, and hence find the exact value of k.
Show all of your algebraic working clearly.

$k =$ ..... *(4 marks)*

(b) Hence, find the coordinates of the point where the line touches the circle.

$(\ldots\ldots\ldots, \ldots\ldots\ldots)$ *(2 marks)*

(Total for Question is 6 marks)

Worked Answers

I Do

Substitute $y = 2x - 3$ into $2x^2 + y^2 = 30$:

$2x^2 + (2x-3)^2 = 30 \implies 2x^2 + 4x^2 - 12x + 9 = 30 \implies 6x^2 - 12x - 21 = 0.$

Divide by 3: $2x^2 - 4x - 7 = 0$.

Use the quadratic formula:

$$x = \frac{4 \pm \sqrt{(-4)^2 - 4(2)(-7)}}{4} = \frac{4 \pm \sqrt{72}}{4} = \frac{4 \pm 6\sqrt{2}}{4} = 1 \pm \frac{3\sqrt{2}}{2}.$$

Find $y = 2x - 3$:

$$y = 2\left(1 \pm \frac{3\sqrt{2}}{2}\right) - 3 = 2 \pm 3\sqrt{2} - 3 = -1 \pm 3\sqrt{2}.$$

Solutions: $\mathbf{x = 1 + \frac{3\sqrt{2}}{2}}$, $\mathbf{y = -1 + 3\sqrt{2}}$ and $\mathbf{x = 1 - \frac{3\sqrt{2}}{2}}$, $\mathbf{y = -1 - 3\sqrt{2}}$.

We Do

Substitute $y = x - 2$ into $x^2 + y^2 = 12$:

$x^2 + (x-2)^2 = 12 \implies 2x^2 - 4x - 8 = 0 \implies x^2 - 2x - 4 = 0.$

$$x = \frac{2 \pm \sqrt{4 + 16}}{2} = \frac{2 \pm 2\sqrt{5}}{2} = 1 \pm \sqrt{5}.$$

$y = (1 \pm \sqrt{5}) - 2 = -1 \pm \sqrt{5}.$

Solutions: $\mathbf{x = 1 + \sqrt{5}}$, $\mathbf{y = -1 + \sqrt{5}}$ and $\mathbf{x = 1 - \sqrt{5}}$, $\mathbf{y = -1 - \sqrt{5}}$.

You Do 1

Rearrange to $y = 2x + 5$ and substitute:

$x^2 + (2x+5)^2 = 25 \implies 5x^2 + 20x + 25 = 25 \implies 5x^2 + 20x = 0 \implies 5x(x+4) = 0.$

Solutions: $\mathbf{x = 0}$, $\mathbf{y = 5}$ and $\mathbf{x = -4}$, $\mathbf{y = -3}$.

You Do 2

Substitute $x = 2y - 2$ into $x^2 + 2y^2 = 18$:

$(2y-2)^2 + 2y^2 = 18 \implies 6y^2 - 8y - 14 = 0 \implies 3y^2 - 4y - 7 = 0 \implies (3y-7)(y+1) = 0.$

When $y = -1$: $x = 2(-1) - 2 = -4$. When $y = \frac{7}{3}$: $x = 2\left(\frac{7}{3}\right) - 2 = \frac{8}{3}$.

Solutions: $\mathbf{x = -4}$, $\mathbf{y = -1}$ and $\mathbf{x = \frac{8}{3}}$, $\mathbf{y = \frac{7}{3}}$.

You Do 3

Substitute $y = 2x - 2$ into $x^2 + y^2 = 20$:

$x^2 + (2x-2)^2 = 20 \implies 5x^2 - 8x - 16 = 0.$

$$x = \frac{8 \pm \sqrt{64 + 320}}{10} = \frac{8 \pm \sqrt{384}}{10} = \frac{8 \pm 8\sqrt{6}}{10} = \frac{4 \pm 4\sqrt{6}}{5}.$$

$$y = 2\left(\frac{4 \pm 4\sqrt{6}}{5}\right) - 2 = \frac{-2 \pm 8\sqrt{6}}{5}.$$

Solutions: $\mathbf{x = \frac{4 + 4\sqrt{6}}{5}}$, $\mathbf{y = \frac{-2 + 8\sqrt{6}}{5}}$ and $\mathbf{x = \frac{4 - 4\sqrt{6}}{5}}$, $\mathbf{y = \frac{-2 - 8\sqrt{6}}{5}}$.

Challenge 1

1. Rearrange $2x + y = 7$ to $y = 7 - 2x$.
2. Substitute into $x^2 + xy = 12$:

$x^2 + x(7 - 2x) = 12 \implies x^2 + 7x - 2x^2 = 12 \implies x^2 - 7x + 12 = 0$.

3. Factorise: $(x - 3)(x - 4) = 0 \implies x = 3$ or $x = 4$.
4. Find y: when $x = 3$, $y = 1$; when $x = 4$, $y = -1$.

Solutions: x = 3, y = 1 and **x = 4, y = −1**.

Challenge 2

(a) Substitute $y = x + 1$ into $x^2 + y^2 = 25$:

$x^2 + (x + 1)^2 = 25 \implies 2x^2 + 2x - 24 = 0 \implies x^2 + x - 12 = 0 \implies (x + 4)(x - 3) = 0$.

$x = -4 \implies y = -3$; $x = 3 \implies y = 4$.

Coordinates: $(-\mathbf{4}, -\mathbf{3})$ and $(\mathbf{3}, \mathbf{4})$.

(b) Use the distance formula:

$AB = \sqrt{\left(3 - (-4)\right)^2 + \left(4 - (-3)\right)^2} = \sqrt{7^2 + 7^2} = \sqrt{98} = \sqrt{49 \times 2} = \mathbf{7\sqrt{2}}$.

Extreme Challenge

(a) Substitute $y = 2x + k$ into $x^2 + y^2 = 20$:

$x^2 + (2x + k)^2 = 20$

$x^2 + 4x^2 + 4kx + k^2 = 20 \implies 5x^2 + 4kx + (k^2 - 20) = 0$. *(Shown)*

Since the line is a tangent, there is exactly one point of intersection, so the discriminant $b^2 - 4ac = 0$:

$(4k)^2 - 4(5)(k^2 - 20) = 0$

$16k^2 - 20k^2 + 400 = 0 \implies 4k^2 = 400 \implies k^2 = 100$.

Since $k > 0$, **k = 10**.

(b) Substitute $k = 10$ back into the quadratic equation:

$5x^2 + 40x + 80 = 0 \implies x^2 + 8x + 16 = 0 \implies (x + 4)^2 = 0 \implies x = -4$.

Find y: $y = 2(-4) + 10 = 2$.

Point of contact: $(-\mathbf{4}, \mathbf{2})$.