Vector Geometry

Practice Worksheet (Problem Solving)

I Do

In triangle OAB, vector $OA = \mathbf{a}$ and vector $OB = \mathbf{b}$.
M is the midpoint of AB. N lies on OB such that $ON : NB = 1 : 2$.
Find vector MN in terms of **a** and **b**, in its simplest form.

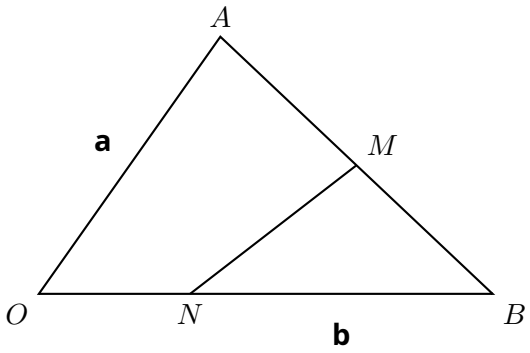


Diagram not to scale

.....

(Total for Question is 4 marks)

We Do

In triangle OAB, $\overrightarrow{OA} =$ **a** and $\overrightarrow{OB} =$ **b**.
M is the midpoint of AB. N lies on OB such that $ON : NB = 3 : 1$.
Find $\overrightarrow{MN}$ in terms of **a** and **b**, in its simplest form.

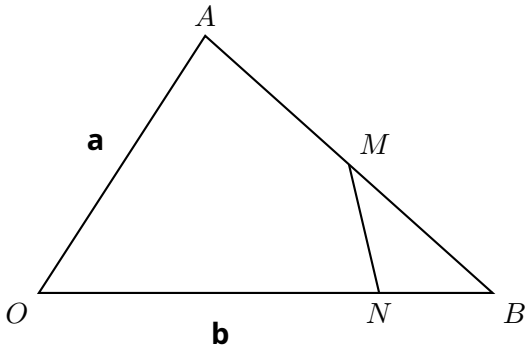


Diagram not to scale

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(Total for Question is 4 marks)

You Do 1

In triangle OAB, $\overrightarrow{OA} =$ **a** and $\overrightarrow{OB} =$ **b**.
P lies on AB such that $AP : PB = 1 : 2$. Q is the midpoint of OB.
Find $\overrightarrow{PQ}$ in terms of **a** and **b**, in its simplest form.

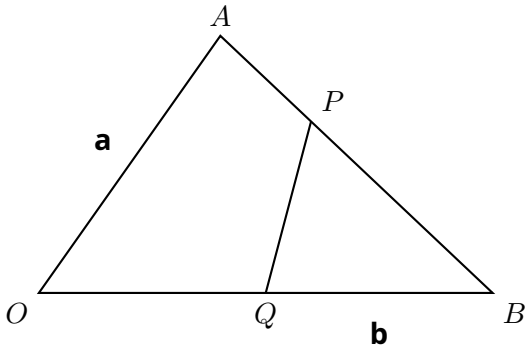


Diagram not to scale

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(Total for Question is 4 marks)

You Do 2

In triangle OAB, $\overrightarrow{OA} = 2\mathbf{a}$ and $\overrightarrow{OB} = 3\mathbf{b}$.
M is the midpoint of OA. N lies on AB such that $AN : NB = 2 : 1$.
Find $\overrightarrow{MN}$ in terms of **a** and **b**, in its simplest form.

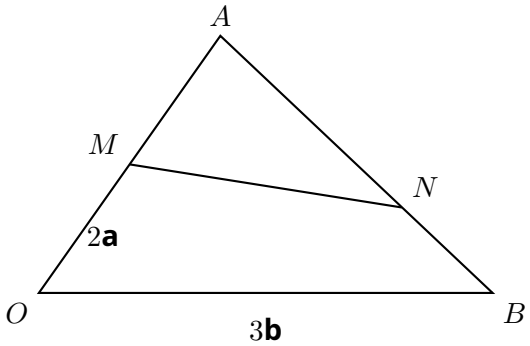


Diagram not to scale

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(Total for Question is 4 marks)

You Do 3

$OABC$ is a parallelogram where $\overrightarrow{OA} =$ **a** and $\overrightarrow{OC} =$ **c**.
M is the midpoint of AB. N lies on BC such that $BN : NC = 1 : 3$.
Find $\overrightarrow{MN}$ in terms of **a** and **c**, in its simplest form.

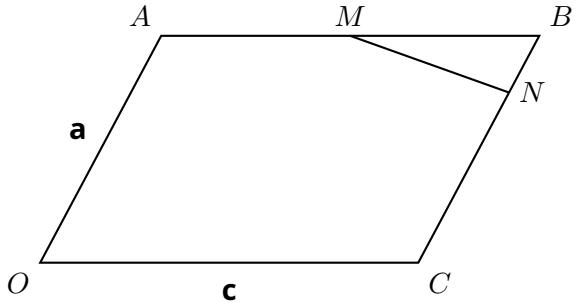


Diagram not to scale

.....

(Total for Question is 4 marks)

Challenge 1

In triangle OAB, $\overrightarrow{OA} = 3\mathbf{a}$ and $\overrightarrow{OB} = 3\mathbf{b}$.
P lies on OA such that $OP : PA = 2 : 1$.
Q lies on OB such that $OQ : QB = 2 : 1$.

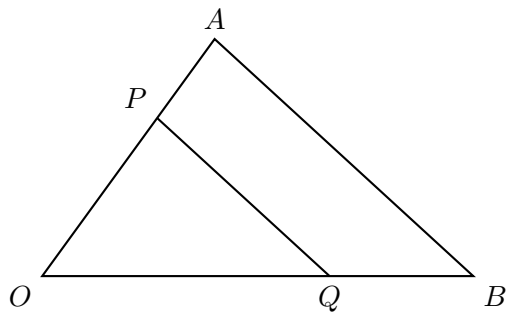


Diagram not to scale

(a) Find $\overrightarrow{PQ}$ in terms of **a** and **b**.

..... *(2 marks)*

(b) Use vectors to prove that PQ is parallel to AB, and write down the ratio $PQ : AB$.

$PQ : AB =$ ..... *(3 marks)*

(Total for Question is 5 marks)

Challenge 2

In triangle OAB, $\overrightarrow{OA} = 2\mathbf{a}$ and $\overrightarrow{OB} = 2\mathbf{b}$.
M is the midpoint of OA.
N lies on AB such that $AN : NB = 2 : 1$.
The line OB is extended to the point C such that $OB = BC$.

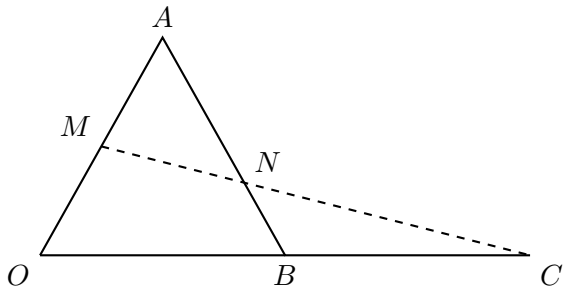


Diagram not to scale

Prove, using vectors, that M, N, and C lie on a straight line.
You must show all of your working clearly.

(Total for Question is 5 marks)

Extreme Challenge

In triangle OAB, $\overrightarrow{OA} =$ **a** and $\overrightarrow{OB} =$ **b**.
P is the midpoint of OA.
Q lies on OB such that $OQ : QB = 2 : 1$.
The straight lines AQ and BP intersect at the point X.

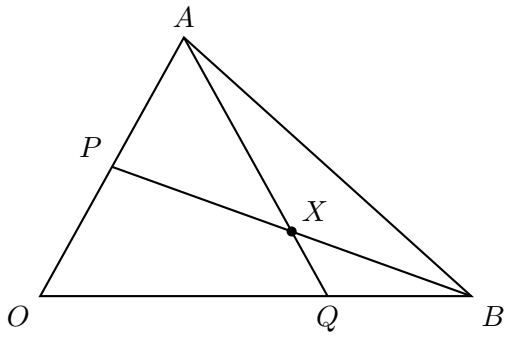


Diagram not to scale

Use a vector method to find the ratio $AX : XQ$.
You must show all of your algebraic working clearly.

$AX : XQ =$ .....

(Total for Question is 5 marks)

Worked Answers

I Do

1. $\overrightarrow{AB} = \overrightarrow{AO} + \overrightarrow{OB} = -\mathbf{a} + \mathbf{b} = \mathbf{b} - \mathbf{a}$.
2. Since M is the midpoint of AB: $\overrightarrow{MB} = \frac{1}{2}(\mathbf{b} - \mathbf{a})$.
3. Since $ON : NB = 1 : 2$, $\overrightarrow{BN} = -\frac{2}{3}\mathbf{b}$.
4. $\overrightarrow{MN} = \overrightarrow{MB} + \overrightarrow{BN} = \frac{1}{2}\mathbf{b} - \frac{1}{2}\mathbf{a} - \frac{2}{3}\mathbf{b} = \mathbf{-\frac{1}{2}a - \frac{1}{6}b}$ *(or* $-\frac{1}{6}(3\mathbf{a} + \mathbf{b})$*)*.

We Do

$\overrightarrow{OM} = \mathbf{a} + \frac{1}{2}(\mathbf{b} - \mathbf{a}) = \frac{1}{2}\mathbf{a} + \frac{1}{2}\mathbf{b}$, $\quad \overrightarrow{ON} = \frac{3}{4}\mathbf{b}$.

$\overrightarrow{MN} = \overrightarrow{ON} - \overrightarrow{OM} = \frac{3}{4}\mathbf{b} - \left(\frac{1}{2}\mathbf{a} + \frac{1}{2}\mathbf{b}\right) = \mathbf{-\frac{1}{2}a + \frac{1}{4}b}$.

You Do 1

$\overrightarrow{OP} = \mathbf{a} + \frac{1}{3}(\mathbf{b} - \mathbf{a}) = \frac{2}{3}\mathbf{a} + \frac{1}{3}\mathbf{b}$, $\quad \overrightarrow{OQ} = \frac{1}{2}\mathbf{b}$.

$\overrightarrow{PQ} = \overrightarrow{OQ} - \overrightarrow{OP} = \frac{1}{2}\mathbf{b} - \frac{2}{3}\mathbf{a} - \frac{1}{3}\mathbf{b} = \mathbf{-\frac{2}{3}a + \frac{1}{6}b}$.

You Do 2

$\overrightarrow{MA} = \mathbf{a}$, $\quad \overrightarrow{AB} = 3\mathbf{b} - 2\mathbf{a}$.

$\overrightarrow{AN} = \frac{2}{3}(3\mathbf{b} - 2\mathbf{a}) = 2\mathbf{b} - \frac{4}{3}\mathbf{a}$.

$\overrightarrow{MN} = \overrightarrow{MA} + \overrightarrow{AN} = \mathbf{a} + 2\mathbf{b} - \frac{4}{3}\mathbf{a} = \mathbf{-\frac{1}{3}a + 2b}$.

You Do 3

In parallelogram $OABC$, $\overrightarrow{AB} = \overrightarrow{OC} = \mathbf{c}$ and $\overrightarrow{BC} = -\overrightarrow{OA} = -\mathbf{a}$.

$\overrightarrow{MB} = \frac{1}{2}\mathbf{c}$, $\quad \overrightarrow{BN} = \frac{1}{4}\overrightarrow{BC} = -\frac{1}{4}\mathbf{a}$.

$\overrightarrow{MN} = \overrightarrow{MB} + \overrightarrow{BN} = \mathbf{\frac{1}{2}c - \frac{1}{4}a}$.

Challenge 1

(a) $\overrightarrow{OP} = \frac{2}{3}(3\mathbf{a}) = 2\mathbf{a}$, $\overrightarrow{OQ} = \frac{2}{3}(3\mathbf{b}) = 2\mathbf{b}$.

$\overrightarrow{PQ} = \overrightarrow{PO} + \overrightarrow{OQ} = \mathbf{2b} - \mathbf{2a}$ *(or* $2(\mathbf{b} - \mathbf{a})$*).*

(b) $\overrightarrow{AB} = \overrightarrow{AO} + \overrightarrow{OB} = 3\mathbf{b} - 3\mathbf{a} = 3(\mathbf{b} - \mathbf{a})$.

Since $\overrightarrow{PQ} = \frac{2}{3}\overrightarrow{AB}$, $\overrightarrow{PQ}$ is a scalar multiple of $\overrightarrow{AB}$, so PQ is parallel to AB.

Ratio $PQ : AB = \mathbf{2} : \mathbf{3}$.

Challenge 2

1. Find $\overrightarrow{MN}$:

$\overrightarrow{MA} = \mathbf{a}$ and $\overrightarrow{AB} = 2\mathbf{b} - 2\mathbf{a}$.

$\overrightarrow{AN} = \frac{2}{3}(2\mathbf{b} - 2\mathbf{a}) = \frac{4}{3}\mathbf{b} - \frac{4}{3}\mathbf{a}$.

$\overrightarrow{MN} = \overrightarrow{MA} + \overrightarrow{AN} = \mathbf{a} + \frac{4}{3}\mathbf{b} - \frac{4}{3}\mathbf{a} = -\frac{1}{3}\mathbf{a} + \frac{4}{3}\mathbf{b} = \frac{1}{3}(4\mathbf{b} - \mathbf{a})$.

2. Find $\overrightarrow{MC}$:

Since $OB = BC$, $\overrightarrow{OC} = 4\mathbf{b}$.

$\overrightarrow{MC} = \overrightarrow{MO} + \overrightarrow{OC} = -\mathbf{a} + 4\mathbf{b} = 4\mathbf{b} - \mathbf{a}$.

3. Conclusion:

$\overrightarrow{MC} = 3\overrightarrow{MN}$, so $\overrightarrow{MC}$ is a scalar multiple of $\overrightarrow{MN}$ (meaning they are parallel). Since both vectors share the common point M, the points M, N, and C lie on a straight line.

Extreme Challenge

1. Express $\overrightarrow{OX}$ in two ways using scalar parameters λ and μ:

- Since X lies on AQ, let $\overrightarrow{AX} = \lambda\overrightarrow{AQ}$, where $\overrightarrow{AQ} = \frac{2}{3}\mathbf{b} - \mathbf{a}$:

$\overrightarrow{OX} = \mathbf{a} + \lambda\left(\frac{2}{3}\mathbf{b} - \mathbf{a}\right) = (1 - \lambda)\mathbf{a} + \frac{2}{3}\lambda\mathbf{b}$.

- Since X lies on BP, let $\overrightarrow{BX} = \mu\overrightarrow{BP}$, where $\overrightarrow{BP} = \frac{1}{2}\mathbf{a} - \mathbf{b}$:

$\overrightarrow{OX} = \mathbf{b} + \mu\left(\frac{1}{2}\mathbf{a} - \mathbf{b}\right) = \frac{1}{2}\mu\mathbf{a} + (1 - \mu)\mathbf{b}$.

2. Since **a** and **b** are non-parallel, equate their coefficients:

- Coefficient of **a**: $1 - \lambda = \frac{1}{2}\mu \implies \mu = 2 - 2\lambda$.

- Coefficient of **b**: $\frac{2}{3}\lambda = 1 - \mu$.

3. Substitute $\mu = 2 - 2\lambda$ into the second equation:

$\frac{2}{3}\lambda = 1 - (2 - 2\lambda) \implies \frac{2}{3}\lambda = 2\lambda - 1 \implies 1 = \frac{4}{3}\lambda \implies \lambda = \frac{3}{4}$.

4. Since $\overrightarrow{AX} = \frac{3}{4}\overrightarrow{AQ}$, X divides AQ in the ratio **AX** : **XQ** $= \mathbf{3} : \mathbf{1}$.