Turning Points & Differentiation

Practice Worksheet (Problem Solving)

I Do

A curve has equation $y = 2x^3 - 3x^2 - 12x + 5$.
Find the coordinates of its turning points. Show clear algebraic working.

(.....,.....) and (.....,.....)

(Total for Question is 5 marks)

We Do

A curve has equation $y = x^3 - 3x^2 - 9x + 7$.
Find the coordinates of its turning points. Show clear algebraic working.

(.....,.....) and (.....,.....)

(Total for Question is 4 marks)

You Do 1

A curve has equation $y = 2x^3 + 3x^2 - 12x + 1$.
Find the coordinates of its turning points. Show clear algebraic working.

(.....,.....) and (.....,.....)

(Total for Question is 4 marks)

You Do 2

A curve has equation $y = -x^3 + 3x^2 + 9x - 4$.

(a) Find the coordinates of the two turning points of the curve. Show clear algebraic working.

$(\ldots\ldots\ldots, \ldots\ldots\ldots)$ and $(\ldots\ldots\ldots, \ldots\ldots\ldots)$ *(4 marks)*

(b) Determine whether each turning point is a maximum or a minimum. Give a reason for your answer.

(2 marks)

(Total for Question is 6 marks)

You Do 3

A curve has equation $y = x(x-3)^2 - 2$.

(a) Expand the right-hand side and find $\frac{dy}{dx}$.

$\frac{dy}{dx} =$ ..... *(2 marks)*

(b) Find the coordinates of the two turning points of the curve, and identify which is a maximum and which is a minimum.

Max: $(\ldots\ldots, \ldots\ldots)$ Min: $(\ldots\ldots, \ldots\ldots)$ *(4 marks)*

(Total for Question is 6 marks)

Challenge 1

A curve has equation

$$y = x^3 + ax^2 + bx + 4$$

where a and b are constants.

The curve has a turning point at $(2, -24)$.

(a) Find the value of a and the value of b.
Show all of your algebraic working clearly.

$a =$ ..... $b =$ ..... *(4 marks)*

(b) Find the coordinates of the other turning point of the curve.

$(\ldots\ldots\ldots, \ldots\ldots\ldots)$ *(2 marks)*

(Total for Question is 6 marks)

Challenge 2

A curve has equation

$$y = x^2 + \frac{16}{x}, \qquad \text{where } x \neq 0.$$

(a) Find the coordinates of the turning point of the curve.
Show all of your algebraic working clearly.

$(\ldots\ldots\ldots, \ldots\ldots\ldots)$ *(4 marks)*

(b) Determine whether this turning point is a maximum or a minimum.
You must show clear working to justify your answer.

(2 marks)

(Total for Question is 6 marks)

Extreme Challenge

A curve has equation $y = x^3 - 3x^2 - 24x + 10$.

(a) Find the coordinates of its two turning points, A and B, where A is the local maximum and B is the local minimum.

$A = (\ldots\ldots, \ldots\ldots)$ $B = (\ldots\ldots, \ldots\ldots)$ *(4 marks)*

(b) Find the equation of the straight line passing through A and B.
Give your answer in the form $y = mx + c$.

..... *(2 marks)*

(c) Write down the range of values of k for which the equation

$$x^3 - 3x^2 - 24x + 10 = k$$

has exactly 3 distinct real roots.

..... *(1 mark)*

(Total for Question is 7 marks)

Worked Answers

I Do

1. Differentiate y with respect to x: $\frac{dy}{dx} = 6x^2 - 6x - 12$.

2. At turning points, $\frac{dy}{dx} = 0$:

$6x^2 - 6x - 12 = 0 \implies x^2 - x - 2 = 0 \implies (x-2)(x+1) = 0 \implies x = 2$ or $x = -1$.

3. Substitute each x-value back into $y = 2x^3 - 3x^2 - 12x + 5$:

- When $x = 2$: $y = 2(2)^3 - 3(2)^2 - 12(2) + 5 = 16 - 12 - 24 + 5 = -15$.
- When $x = -1$: $y = 2(-1)^3 - 3(-1)^2 - 12(-1) + 5 = -2 - 3 + 12 + 5 = 12$.

Turning points: $(\mathbf{2}, \mathbf{-15})$ **and** $(\mathbf{-1}, \mathbf{12})$.

We Do

$\frac{dy}{dx} = 3x^2 - 6x - 9 = 0 \implies 3(x-3)(x+1) = 0 \implies x = 3$ or $x = -1$.

- When $x = 3$: $y = 27 - 27 - 27 + 7 = -20$.
- When $x = -1$: $y = -1 - 3 + 9 + 7 = 12$.

Turning points: $(\mathbf{3}, \mathbf{-20})$ **and** $(\mathbf{-1}, \mathbf{12})$.

You Do 1

$\frac{dy}{dx} = 6x^2 + 6x - 12 = 0 \implies 6(x+2)(x-1) = 0 \implies x = -2$ or $x = 1$.

- When $x = -2$: $y = 2(-8) + 3(4) - 12(-2) + 1 = -16 + 12 + 24 + 1 = 21$.
- When $x = 1$: $y = 2 + 3 - 12 + 1 = -6$.

Turning points: $(\mathbf{-2}, \mathbf{21})$ **and** $(\mathbf{1}, \mathbf{-6})$.

You Do 2

(a) $\frac{dy}{dx} = -3x^2 + 6x + 9 = 0 \implies -3(x^2 - 2x - 3) = 0 \implies (x-3)(x+1) = 0 \implies x = 3$ or $x = -1$.

- When $x = 3$: $y = -27 + 27 + 27 - 4 = 23$.
- When $x = -1$: $y = 1 + 3 - 9 - 4 = -9$.

Turning points: $(\mathbf{3}, \mathbf{23})$ **and** $(\mathbf{-1}, \mathbf{-9})$.

(b) $\frac{d^2y}{dx^2} = -6x + 6$.

- At $x = 3$: $\frac{d^2y}{dx^2} = -12 < 0 \implies (\mathbf{3}, \mathbf{23})$ **is a maximum**.
- At $x = -1$: $\frac{d^2y}{dx^2} = 12 > 0 \implies (\mathbf{-1}, \mathbf{-9})$ **is a minimum**.

You Do 3

(a) $y = x(x^2 - 6x + 9) - 2 = x^3 - 6x^2 + 9x - 2 \implies \mathbf{\frac{dy}{dx} = 3x^2 - 12x + 9}$.

(b) $3(x^2 - 4x + 3) = 0 \implies (x-1)(x-3) = 0 \implies x = 1$ or $x = 3$.

- When $x = 1$: $y = 1(-2)^2 - 2 = 2$; $\frac{d^2y}{dx^2} = 6(1) - 12 = -6 < 0 \implies$ **Maximum at** $(\mathbf{1}, \mathbf{2})$.
- When $x = 3$: $y = 3(0)^2 - 2 = -2$; $\frac{d^2y}{dx^2} = 6(3) - 12 = 6 > 0 \implies$ **Minimum at** $(\mathbf{3}, \mathbf{-2})$.

Challenge 1

(a) 1. Since $(2, -24)$ lies on the curve:
$-24 = 2^3 + a(2)^2 + b(2) + 4 \implies -24 = 12 + 4a + 2b \implies 4a + 2b = -36 \implies 2a + b = -18.$
2. Differentiate: $\frac{dy}{dx} = 3x^2 + 2ax + b$. Since $x = 2$ is a turning point, $\frac{dy}{dx} = 0$ when $x = 2$:
$3(2)^2 + 2a(2) + b = 0 \implies 12 + 4a + b = 0 \implies 4a + b = -12.$
3. Subtract the first equation from the second:
$(4a + b) - (2a + b) = -12 - (-18) \implies 2a = 6 \implies$ **a** $= 3$.
4. Substitute $a = 3$ into $2a + b = -18$: $6 + b = -18 \implies$ **b** $= -24$.

(b) $\frac{dy}{dx} = 3x^2 + 6x - 24 = 0 \implies 3(x - 2)(x + 4) = 0 \implies x = -4.$
When $x = -4$: $y = (-4)^3 + 3(-4)^2 - 24(-4) + 4 = -64 + 48 + 96 + 4 = 84.$
Other turning point: $(-$**4**$,$ **84**$)$.

Challenge 2

(a) Rewrite using negative indices: $y = x^2 + 16x^{-1}$.
Differentiate: $\frac{dy}{dx} = 2x - 16x^{-2} = 2x - \frac{16}{x^2}$.
Set $\frac{dy}{dx} = 0$:
$2x = \frac{16}{x^2} \implies 2x^3 = 16 \implies x^3 = 8 \implies x = 2.$
When $x = 2$: $y = 2^2 + \frac{16}{2} = 4 + 8 = 12.$
Turning point: $($**2**$,$ **12**$)$.

(b) $\frac{d^2y}{dx^2} = 2 + 32x^{-3} = 2 + \frac{32}{x^3}$.
When $x = 2$: $\frac{d^2y}{dx^2} = 2 + \frac{32}{8} = 2 + 4 = 6 > 0.$
Since $\frac{d^2y}{dx^2} > 0$, $(2, 12)$ is a **local minimum**.

Extreme Challenge

(a) $\frac{dy}{dx} = 3x^2 - 6x - 24 = 0 \implies 3(x^2 - 2x - 8) = 0 \implies 3(x - 4)(x + 2) = 0 \implies x = -2$ or $x = 4.$
Second derivative: $\frac{d^2y}{dx^2} = 6x - 6.$
- At $x = -2$: $y = (-2)^3 - 3(-2)^2 - 24(-2) + 10 = -8 - 12 + 48 + 10 = 38$; $\frac{d^2y}{dx^2} = -18 < 0 \implies$ **A** $= (-$**2**$,$ **38**$)$ (maximum).
- At $x = 4$: $y = 4^3 - 3(4)^2 - 24(4) + 10 = 64 - 48 - 96 + 10 = -70$; $\frac{d^2y}{dx^2} = 18 > 0 \implies$ **B** $= ($**4**$, -$**70**$)$ (minimum).

(b) Gradient of AB: $m = \frac{-70 - 38}{4 - (-2)} = \frac{-108}{6} = -18.$
Using $y = -18x + c$ with $(-2, 38)$: $38 = -18(-2) + c \implies 38 = 36 + c \implies c = 2.$
Equation of line AB: **y** $= -$**18x** $+$ **2**.

(c) The horizontal line $y = k$ intersects the cubic curve at 3 distinct points when k lies strictly between the y-coordinates of the local minimum and local maximum:
$-$**70** $<$ **k** $<$ **38**.