Functions and Inverses

Practice Worksheet (Problem Solving)

I Do

The functions f and g are defined by

$$f(x) = \frac{2x+1}{x-3} \quad (x \neq 3), \quad g(x) = x^2 - 4$$

(a) Find $f(4)$.

. . . . . *(1 mark)*

(b) Solve the equation $fg(x) = \frac{11}{2}$.

. . . . . *(4 marks)*

(c) Find the inverse function $f^{-1}(x)$.

. . . . . *(2 marks)*

(Total for Question is 7 marks)

Video walkthrough

We Do

The functions f and g are defined by

$$f(x) = \frac{3x-1}{x-2} \quad (x \neq 2), \quad g(x) = x^2 - 3$$

(a) Find $f(3)$.

.....

(1 mark)

(b) Solve the equation $fg(x) = 4$.

.....

(4 marks)

(c) Find the inverse function $f^{-1}(x)$.

.....

(2 marks)

(Total for Question is 7 marks)

You Do 1

The functions f and g are defined by

$$f(x) = \frac{4x+3}{x-1} \quad (x \neq 1), \quad g(x) = x^2 - 2$$

(a) Find $f(2)$.

.....

(1 mark)

(b) Solve the equation $fg(x) = 5$.

.....

(4 marks)

(c) Find the inverse function $f^{-1}(x)$.

.....

(2 marks)

(Total for Question is 7 marks)

You Do 2

The functions f and g are defined by

$$f(x) = \frac{x+5}{2x-3} \quad (x \neq 1.5), \quad g(x) = x^2 - 2$$

(a) Find $f(4)$.

.....

(1 mark)

(b) Solve the equation $fg(x) = 1$.

.....

(4 marks)

(c) Find the inverse function $f^{-1}(x)$.

.....

(2 marks)

(Total for Question is 7 marks)

You Do 3

The functions f and g are defined by

$$f(x) = \frac{5-x}{x+2} \quad (x \neq -2), \quad g(x) = 2x^2 + 3$$

(a) Find $f(1)$.

.....

(1 mark)

(b) Solve the equation $fg(x) = 0$.

.....

(4 marks)

(c) Find the inverse function $f^{-1}(x)$.

.....

(2 marks)

(Total for Question is 7 marks)

Challenge 1

The functions f and g are defined by

$$f(x) = \frac{3x+2}{x-1} \quad (x \neq 1), \quad g(x) = 2x - 5$$

(a) Find the inverse function $f^{-1}(x)$.

. . . . . *(2 marks)*

(b) Solve the equation $f^{-1}(x) = g(x)$.
Give your answer in the form $\frac{a \pm \sqrt{b}}{c}$ where a, b, and c are integers.

. . . . . *(5 marks)*

(Total for Question is 7 marks)

Challenge 2

The function h is defined by

$$h(x) = \frac{ax+4}{x-a} \quad (x \neq a)$$

where a is a constant.

(a) Prove that $h^{-1}(x) = h(x)$ for all $x \neq a$.

. . . . . *(3 marks)*

(b) Given that $h(3) = 11$, find the exact value of a.

$a =$ . . . . . *(3 marks)*

(Total for Question is 6 marks)

Extreme Challenge

The functions f and g are defined by

$$f(x) = \frac{2x}{x-1} \quad (x \neq 1), \quad g(x) = \frac{x+1}{x} \quad (x \neq 0)$$

(a) Find an expression for $fg(x)$.
Give your answer in its simplest form.

.....

(3 marks)

(b) Solve the equation $gf(x) = 3$.

.....

(4 marks)

(Total for Question is 7 marks)

Answers & Worked Solutions

I Do

(a) $f(4) = \frac{2(4)+1}{4-3} = \frac{9}{1} = \mathbf{9}$.

(b) $fg(x) = f(x^2-4) = \frac{2(x^2-4)+1}{(x^2-4)-3} = \frac{2x^2-8+1}{x^2-7} = \frac{2x^2-7}{x^2-7}$.

$\frac{2x^2-7}{x^2-7} = \frac{11}{2}$

$2(2x^2-7) = 11(x^2-7)$

$4x^2-14 = 11x^2-77$

$63 = 7x^2 \implies x^2 = 9 \implies \mathbf{x = \pm 3}$.

(c) Let $y = \frac{2x+1}{x-3}$.

$y(x-3) = 2x+1 \implies xy-3y = 2x+1$

$xy-2x = 3y+1 \implies x(y-2) = 3y+1 \implies x = \frac{3y+1}{y-2}$.

$\mathbf{f^{-1}(x) = \frac{3x+1}{x-2}}$.

We Do

(a) $f(3) = \frac{3(3)-1}{3-2} = \frac{8}{1} = \mathbf{8}$.

(b) $fg(x) = f(x^2-3) = \frac{3(x^2-3)-1}{(x^2-3)-2} = \frac{3x^2-9-1}{x^2-5} = \frac{3x^2-10}{x^2-5}$.

$\frac{3x^2-10}{x^2-5} = 4 \implies 3x^2-10 = 4(x^2-5) \implies 3x^2-10 = 4x^2-20$

$10 = x^2 \implies \mathbf{x = \pm\sqrt{10}}$.

(c) $y = \frac{3x-1}{x-2} \implies y(x-2) = 3x-1 \implies xy-2y = 3x-1$

$xy-3x = 2y-1 \implies x(y-3) = 2y-1 \implies x = \frac{2y-1}{y-3}$.

$\mathbf{f^{-1}(x) = \frac{2x-1}{x-3}}$.

You Do 1

(a) $f(2) = \frac{4(2)+3}{2-1} = \frac{11}{1} = \mathbf{11}$.

(b) $fg(x) = f(x^2-2) = \frac{4(x^2-2)+3}{(x^2-2)-1} = \frac{4x^2-8+3}{x^2-3} = \frac{4x^2-5}{x^2-3}$.

$\frac{4x^2-5}{x^2-3} = 5 \implies 4x^2-5 = 5(x^2-3) \implies 4x^2-5 = 5x^2-15$

$10 = x^2 \implies \mathbf{x = \pm\sqrt{10}}$.

(c) $y = \frac{4x+3}{x-1} \implies xy-y = 4x+3 \implies xy-4x = y+3$

$x(y-4) = y+3 \implies x = \frac{y+3}{y-4}$.

$\mathbf{f^{-1}(x) = \frac{x+3}{x-4}}$.

You Do 2

(a) $f(4) = \frac{4+5}{2(4)-3} = \frac{9}{5} = \mathbf{1.8}$.

(b) $fg(x) = f(x^2-2) = \frac{(x^2-2)+5}{2(x^2-2)-3} = \frac{x^2+3}{2x^2-4-3} = \frac{x^2+3}{2x^2-7}$.

$\frac{x^2+3}{2x^2-7} = 1 \implies x^2+3 = 2x^2-7 \implies 10 = x^2 \implies \mathbf{x = \pm\sqrt{10}}$.

(c) $y = \frac{x+5}{2x-3} \implies y(2x-3) = x+5 \implies 2xy-3y = x+5$

$2xy-x = 3y+5 \implies x(2y-1) = 3y+5 \implies x = \frac{3y+5}{2y-1}$.

$\mathbf{f^{-1}(x) = \frac{3x+5}{2x-1}}$.

You Do 3

(a) $f(1) = \frac{5-1}{1+2} = \mathbf{\frac{4}{3}}$.

(b) $fg(x) = f(2x^2+3) = \frac{5-(2x^2+3)}{(2x^2+3)+2} = \frac{2-2x^2}{2x^2+5}$.

$\frac{2-2x^2}{2x^2+5} = 0 \implies 2-2x^2 = 0 \implies 2x^2 = 2 \implies x^2 = 1 \implies \mathbf{x = \pm 1}$.

(c) $y = \frac{5-x}{x+2} \implies y(x+2) = 5-x \implies xy+2y = 5-x$

$xy + x = 5 - 2y \implies x(y+1) = 5-2y \implies x = \frac{5-2y}{y+1}$.

$\mathbf{f^{-1}(x)} = \mathbf{\frac{5-2x}{x+1}}$.

Challenge 1

(a) $y = \frac{3x+2}{x-1} \implies xy - y = 3x + 2 \implies xy - 3x = y + 2$

$x(y-3) = y+2 \implies \mathbf{f^{-1}(x)} = \mathbf{\frac{x+2}{x-3}}$.

(b) $\frac{x+2}{x-3} = 2x - 5$

$x + 2 = (2x-5)(x-3) \implies x + 2 = 2x^2 - 6x - 5x + 15$

$x + 2 = 2x^2 - 11x + 15 \implies 0 = 2x^2 - 12x + 13.$

Using quadratic formula: $x = \frac{-(-12) \pm \sqrt{(-12)^2 - 4(2)(13)}}{2(2)} = \frac{12 \pm \sqrt{144-104}}{4} = \frac{12 \pm \sqrt{40}}{4}$

$\sqrt{40} = 2\sqrt{10}$, so $x = \frac{12 \pm 2\sqrt{10}}{4} = \mathbf{\frac{6 \pm \sqrt{10}}{2}}$.

Challenge 2

(a) To find the inverse, let $y = \frac{ax+4}{x-a}$.

$y(x-a) = ax + 4 \implies xy - ay = ax + 4 \implies xy - ax = ay + 4$

$x(y-a) = ay + 4 \implies x = \frac{ay+4}{y-a}$.

Therefore, $h^{-1}(x) = \frac{ax+4}{x-a}$.

This is exactly the same as the original function $h(x)$, so $h^{-1}(x) = h(x)$.

(b) $h(3) = 11 \implies \frac{3a+4}{3-a} = 11$

$3a + 4 = 11(3-a) \implies 3a + 4 = 33 - 11a$

$14a = 29 \implies \mathbf{a = \frac{29}{14}}$.

Extreme Challenge

(a) $fg(x) = f(\frac{x+1}{x}) = \frac{2(\frac{x+1}{x})}{\frac{x+1}{x} - 1} = \frac{\frac{2x+2}{x}}{\frac{x+1-x}{x}} = \frac{\frac{2x+2}{x}}{\frac{1}{x}}$.

Multiplying numerator and denominator by x gives: $\mathbf{2x + 2}$.

(b) First find $gf(x)$:

$gf(x) = g(\frac{2x}{x-1}) = \frac{\frac{2x}{x-1}+1}{\frac{2x}{x-1}} = \frac{\frac{2x+x-1}{x-1}}{\frac{2x}{x-1}} = \frac{\frac{3x-1}{x-1}}{\frac{2x}{x-1}} = \frac{3x-1}{2x}$.

Now solve $gf(x) = 3$:

$\frac{3x-1}{2x} = 3 \implies 3x - 1 = 6x \implies -1 = 3x \implies \mathbf{x = -\frac{1}{3}}$.